GENERALIZED EIGENVALUE PROBLEMS

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OVERVIEW

- Basics
- Generalized Schur Form
- Shifts / Generalized Shifts
- Partial Generalized Schur Form
- Jacobi-Davidson QZ Algorithm

BASICS

Generalized Eigenvalue Problem $Ax = \lambda Bx$ or $\left(A - \lambda B\right)x = 0$

We call (λ, x) a (right) eigenpair and $\left(A, B\right)$ a (matrix) pencil

Left eigenpair $y^* A = \lambda y^* B \Leftrightarrow A^* y = \bar{\lambda} B^* y$

Pencil $\left(A, B\right)$ regular if $\det\left(A - \lambda B\right)$ is not identically zero (for all λ)

Example singular/degenerate pencil: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Pencil is singular as $\det\left(A - \lambda B\right) = 0$ for any λ.

Regular pencil has finite number of eigenvalues – characteristic equation is polynomial of degree $m \leq n$. If polynomial not identically zero, then at most m zeros. For singular B possible that char. polynomial is nonzero constant.

BASICS

Regular pencil has finite number of eigenvalues

Characteristic polynomial $p(\lambda) = \det(A - \lambda B)$ is poly of degree $m \leq n$

If polynomial not identically zero, then at most m zeros

For singular B possible that characteristic polynomial is nonzero constant – no eigenvalues

If $\lambda \neq 0$ eigenvalue of (A, B) then $\mu = \lambda^{-1}$ eigenvalue of (B, A)

If B singular and $Bx = 0\,(x \neq 0)$, then $(0, x)$ eigenpair of (B, A)

Hence natural to consider (∞, x) eigenpair of (A, B)

Moreover, if polynomial degenerate (not of full degree) then some eigenvalues must be infinite (and B must be singular)

GENERALIZED SCHUR FORM

Generalized Schur form for regular (A, B)

Unitary U, V exist such that $S = U^* A V, T = U^* B V$ both uppertriangular

Generalized eigenvalues: $\det(S - \lambda T) = 0 \Leftrightarrow \lambda = \frac{s_{ii}}{t_{ii}}$ and $\lambda(T, S) = \frac{t_{ii}}{s_{ii}}$

Easy for triangular problem – note better to think of $\langle s_{ii}, t_{ii} \rangle$ than λ

Eigenvalues of (S, T) are eigenvalues of (A, B):

$$(A - \lambda B)x = 0 \Leftrightarrow (A - \lambda B)VV^{-1}x = 0 \Leftrightarrow U^*(A - \lambda B)V(V^{-1}x) = 0$$

Eigenvalue λ unchanged and eigenvector transformed to $V^{-1}x$

Same for left eigenvector $y \rightarrow U^{-1}y$

Prefer unitary U, V for numerical stability

GENERALIZED SCHUR FORM

Note that $\det\left(U^{*}\left(A-\lambda B\right)V\right)=\det\left(U\right)\det\left(V\right)\det\left(A-\lambda B\right)$

Nonsingular U,V do not change the zeros, so eigenvalues and multiplicities are preserved.

For unitary U,V determinants are unit scalars

$p_{(A,B)}\left(\lambda\right)=\det\left(A-\lambda B\right)$ is characteristic polynomial
Algebraic multiplicity of eigenvalue is its multiplicity as root of char. pol.

If m degree of $p_{(A,B)}\left(\lambda\right)$, then $\left(A,B\right)$ has infinite eigenvalue of mult. $n-m$

SHIFTING THE PROBLEM

If (A, B) regular and B nonsingular: $Ax = \lambda Bx \Leftrightarrow B^{-1}Ax = \lambda x$

Useful both for analysis and algorithms.
For singular B (and efficient algorithms) we may want to shift the problem

Take $\mu = \dfrac{\lambda}{1+\omega\lambda} \Leftrightarrow \mu = \lambda - \mu\omega\lambda$

$$Ax = \lambda Bx \Leftrightarrow Ax - \mu\omega Ax = \mu Bx \Leftrightarrow Ax = \mu\left(B + \omega A\right)x$$

If $(B+\omega A)$ is well-conditioned we can work with $\left(B+\omega A\right)^{-1}Ax = \mu x$

In practice, near singular B is just as problematic as singular B

If $(A, B) \rightarrow (\lambda, x)$ then $(A, B+\omega A) \rightarrow \left(\dfrac{\lambda}{1+\omega\lambda}, x\right)$

MORE GENERAL SHIFT

We can shift more generally.

If A and $\begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}$ nonsingular, and

let $\left(C, D\right) = \left(w_{11}A + w_{21}B, w_{12}A + w_{22}B\right)$

Then $\left(\left\langle \alpha, \beta \right\rangle, x\right)$ is eigenpair of $\left(A, B\right)$

if $\left(\left\langle w_{11}\alpha + w_{21}\beta, w_{12}\alpha + w_{22}\beta \right\rangle, x\right)$ eigenpair of $\left(C, D\right)$

Can be used to make methods that require inverse (in some way) of one of the matrices better behaved.

Can be used to base analysis of generalized problem on analysis of standard problem.

PARTIAL GENERALIZED SCHUR FORM

For large problems we are typically interested only in selected eigenvalues and corresponding right (and/or left) eigenspace: *partial generalized Schur form*.

Consider $Ax = \lambda Bx = \alpha/\beta\, Bx \Leftrightarrow \beta Ax - \alpha Bx = 0$

Partial generalized Schur form: Find $Q_k, Z_k \in \mathbb{C}^{n\times k}$ with orthonormal cols and $R_k^A, R_k^B \in \mathbb{C}^{k\times k}$ upper triangular such that $AQ_k = R_k^A$ and $BQ_k = Z_k R_k^B$.
Let $\alpha_i = \left(R_k^A\right)_{ii}$ and $\beta_i = \left(R_k^B\right)_{ii}$ be diagonal coefficients

If $\left(\left\langle \alpha_i, \beta_i \right\rangle, y\right)$ is generalized eigenpair of $\left(R_k^A, R_k^B\right)$, then
$\left(\left\langle \alpha_i, \beta_i \right\rangle, Q_k y\right)$is generalized eigenpair of $\left(A, B\right)$

Note that solving $\left(R_k^A - \alpha_i/\beta_i\, R_k^B\right)y = 0$ is simple (upper triangular)

JACOBI DAVIDSON QZ

Assume we have approximate space V for Q_k and W for Z_k, we use (so-called) Petrov-Galerkin approximation: Find $u \in \text{range}(V) \rightarrow u = Vy$ such that

$$\eta A u - \zeta B u \perp W \Rightarrow \eta W^* A V y - \zeta W^* A V y = 0$$

So, given V, W we solve a small generalized eigenvalue problem (LAPACK) for $\left(W^* A V, W^* B V\right)$ (typically using, dense, standard QZ algorithm)

Find unitary S_L, S_R such that

$$S_L^* \left(W^* A V\right) S_R = T_A \rightarrow \zeta_i \text{ (diag)}$$
$$S_L^* \left(W^* B V\right) S_R = T_B \rightarrow \zeta_i \text{ (diag)}$$

Now VS_R approximates Q_k and WS_L approximates Z_k

We can take W s.t. $\text{range}(W) = \text{range}\left(\nu_0 A V + \mu_0 B V\right)$

JACOBI-DAVIDSON QZ

Improve by solving JD correction equation for $t \perp u$ for pencil $\eta A - \zeta B$

$$\left(I - \frac{pp^*}{p^*p}\right)(\eta A - \zeta B)(I - uu^*)t = -r$$

where

$$r = (\eta A - \zeta B)u$$

$$p = \nu_0 Au + \mu_0 Bu$$

Solve correction equation (very) approximately: t
(exact solve yields quadratic convergence near a solution)
Extend spaces:

$$v_{m+1} = (t - V_m V_m^* t) / \left\| t - V_m V_m^* t \right\|_2 \quad \text{(accurately)}$$

$$\tilde{w} = \nu_0 A v_m + \mu_0 B v_m \text{ and } w_{m+1} = (\tilde{w} - W_m W_m^* \tilde{w}) / \left\| (\tilde{w} - W_m W_m^* \tilde{w}) \right\|_2$$

Note if $u = V(S_R)_1 = VS_R e_1$ then $p = WS_L e_1$

DEFLATION AND RESTART

We have $AQ_{k-1} = Z_{k-1}R^A_{k-1}$ and $BQ_{k-1} = Z_{k-1}R^B_{k-1}$

Extend

$$A\left[Q_{k-1}\ q\right] = \left[Z_{k-1}\ z\right]\begin{bmatrix} R^A_{k-1} & a \\ 0 & \alpha \end{bmatrix} \quad \text{and} \quad B\left[Q_{k-1}\ q\right] = \left[Z_{k-1}\ z\right]\begin{bmatrix} R^B_{k-1} & b \\ 0 & \beta \end{bmatrix}$$

New generalized Schur pair $\left(\left\langle \alpha, \beta \right\rangle, q\right)$:

$$Q^*_{k-1}q = 0 \quad \text{and} \quad \left(I - Z_{k-1}Z^*_{k-1}\right)\left(\beta A - \alpha B\right)\left(I - Q_{k-1}Q^*_{k-1}\right)q = 0$$

where $a = Z^*_{k-1}Aq$ and $b = Z^*_{k-1}Bq$

Hence $\left(\left\langle \alpha, \beta \right\rangle, q\right)$ is eigenpair of

$$\left(\left(I - Z_{k-1}Z^*_{k-1}\right)A\left(I - Q_{k-1}Q^*_{k-1}\right), \left(I - Z_{k-1}Z^*_{k-1}\right)B\left(I - Q_{k-1}Q^*_{k-1}\right)\right)$$

Restarting by using Generalize Schur Form and taking desired part

GOOD READING